A Poetry of Angles and Dreams

by Catherine Asaro

Spaces of the Imagination

The first time I heard about Riemann surfaces, I fell in love with the subject. It was during a course in applied math for physics majors that I took as an undergraduate. I was intimidated by the course, but it looked intriguing, too, so I gave it a try. I adored that class.To this day, applied math remains my favorite subject. Give me an equation to solve, and I'm happy. This essay describes some of the ways I've incorporated math into my stories. I've started out with a few equations for those who enjoy them, but it isn't necessary to understand those to follow the article. I've also included analogies and pictures that I hope will elucidate the beautiful concepts behind the mathematics. My introduction to Riemann sheets came about in that long-ago math class when we delved into the subject of complex analysis. It's all about complex numbers, a subject seen by students of all ages, from preteens first learning about imaginary numbers to doctoral candidates studying theoretical physics. So what is a complex number? We can call it z, where

$$z = x + iy$$

Here x and y are real numbers, that is, numbers such as 42, 3.64, 84/7, or π. However, i is a different beast altogether; it's an imaginary number, specifically the square root of -1:

$$i = \sqrt{-1}$$

So z has a "real part" equal to x and an "imaginary part" equal to iy. If we plot z on a graph of the x-y plane, the point $(x, y) = z$ is the imaginary number.

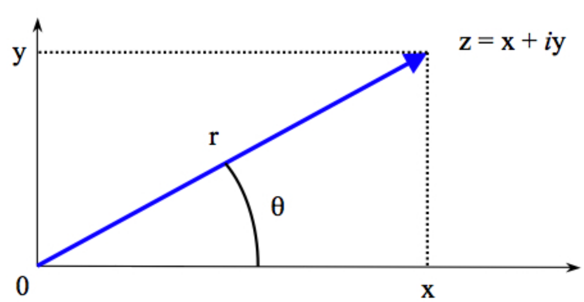

Figure 1: The tip of the arrow gives the complex number $z = x + iy$, where x is the real part and iy is the imaginary part.

We can also represent z by *polar coordinates.* Its position is again given by two numbers, but in this case the two numbers are r and θ. The line drawn from the origin to z has length r, and the angle it makes with the x axis is θ. Polar and x-y coordinates are related; $x = r\cos\theta$, and $y = r\sin\theta$. So

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

where $e^{i\theta} = \cos\theta + i\sin\theta$ is the complex *exponential function*, which shows up in many physics classes, where it has bedeviled numerous generations of incipient young scientists.

The length of r doesn't really matter in these discussions, so to make life easier, I'll set $r = 1$. The angle θ can still vary. In fact, if we let it increase from 0 to 2π, the point z will rotate all the way around the plane and come back to where it started; it makes a circle in the x-y plane.

Imagine that the line from the origin to z is the hour hand on a clock, with z at its tip. When the hand moves once around the clock face, that's analogous to θ moving through a total angle of 2π. After twelve hours, midnight until noon, the hand is back where it started. Go around a second time, and the hour goes from noon to midnight. The hours in the first go-around have the same numbers as in the second one, but refer to different times. Go around a third time, though, and we're back to the morning hours.

The convention in math, however, is that $\theta = 0$ when z is on the x-axis, which corresponds to an hour hand at 3 o'clock. Also, θ increases as z goes counter-clockwise around the plane. If z goes clockwise, θ decreases from 0 to -2π. The discussion is essentially the same, though, regardless of whether or not we have a minus in front of the angles. And we can just as easily go from 3 am to 3 pm as from midnight to noon. So z is a clock. Sort of. At first glance, it has very little to do with fiction. But something amazing happens when we take its square root. Then we get

$$\sqrt{z} = z^{1/2} = e^{i\theta/2} = \cos(\theta/2) + i\sin(\theta/2)$$

What happens now if the "hour hand" moves clockwise? When θ goes all the way around the clock, $\sqrt{z}$ only gets halfway around, because it depends on $\theta/2$. To see what that means, let's look at $\theta = -\pi$, which corresponds to 9 o'clock. For that angle,

$$\sqrt{z} = \cos(-\pi/2) + i\sin(-\pi/2) = -i$$

If we then go around the clock and come back to where we started, that changes θ by -2π, which means we'll be at $\theta = -3\pi$, since we started at $\theta = -\pi$. That's also 9 o'clock. So if $\sqrt{z}$ were well-behaved, it would have the same value at -3π as it did at $-\pi$. However, instead we get

$$\sqrt{z} = \cos(-3\pi/2) + i\sin(-3\pi/2) = i$$

The square root has different values for $-\pi$ and -3π even though they are in exactly the same place, the same "hour." So is $\sqrt{z} = i$ or $-i$? It's ambiguous. That's why double-valued functions aren't allowed; z must be unique at every point to be considered a valid function.

You might wonder what happens if we go around a third time. Is $\sqrt{z}$ triple-valued? Quadruple valued? Where does it stop? As it turns out, the third go-around gives $\sqrt{z} = -i$ again and a fourth gives $\sqrt{z} = i$. So $\sqrt{z}$ alternates between only two values. Unfortunately, even two is too many. In our universe, the math that describes physics requires single-valued quantities. They give unambiguous results; otherwise, we wouldn't know which number to use. But terms like $\sqrt{z}$ come up all the time in the equations of physics. So we seem to be stuck.

The solution to this conundrum is an elegant idea developed by the mathematician Bernhard Riemann in the nineteenth century. Instead of one x-y plane, he suggested stacking two of them together. The top plane, or "sheet," is for points where z has its first value, and the bottom is for its second value. To get from one sheet to another, we slit them from the origin out to infinity. That's called a *branch cut.* We connect the sheets by connecting their branch cuts. Then we can go around the top sheet once and slip through to the bottom for the second time around. Then back to the top sheet. That allows $\sqrt{z}$ to have one value on the top and a different one on the bottom.

Voila! The function is no longer double-valued. The ambiguity goes away as long as we know which sheet we're on. It's like stacking two clocks. The hour hand goes around from 3 am to 3 pm on the top clock, then slides through the branch cut and goes around on the second clock from 3 pm to 3 am. Then back to the top clock.

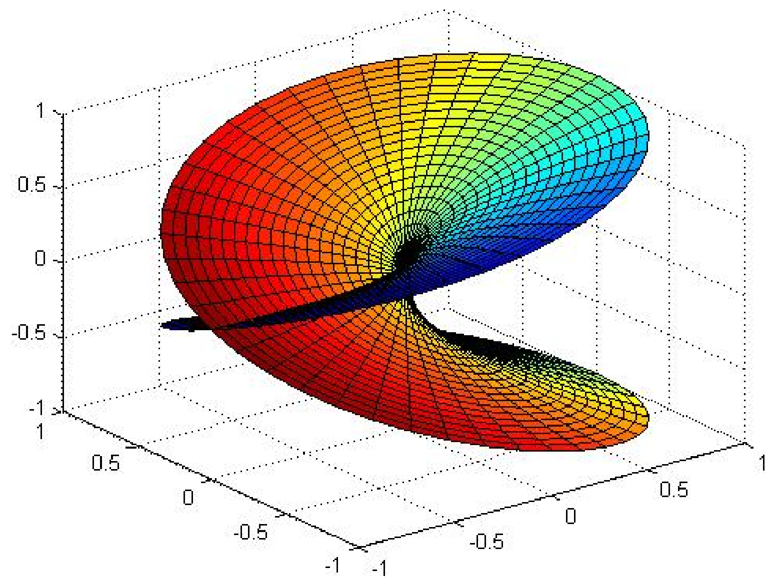

Figure 2: Riemann sheets for $\sqrt{z}$. This shows curved surfaces, which are hardly clock-like. However, the sheets can also be plotted to look flatter, more like disks.

The square root of z is the simplest case of Riemann sheets. If we have a cube root, we need three sheets; a quartic root requires four sheets, and so on for more complicated functions. However, the basic concept remains the same: the sheets create alternate versions of the complex plane.

As a student, I was fascinated by all this math; as a science fiction writer, I was delighted. What a great way to describe alternate universes! Put them on different Riemann sheets. In "The Spacetime Pool," the character Janelle stumbles through a branch cut that drops her into an alternate reality. In my novel *Catch the Lightning*, the fighter pilot Althor is thrown through a rip in the Riemann sheet of his universe when his ship is sabotaged.

The seeds of these ideas in my stories go back to when I was trying to think of a fictional faster-than-light drive that was at least mathematically plausible. I figured out that making speed complex in the equations of special relativity would get rid of the problems with the speed of light. Of course, it's a math game; we know of no physical way to make our speed complex. But the math is pretty, so I wrote a paper about it for *The American Journal of Physics* titled "Complex Speeds and Special Relativity," which appeared in volume 64, the April 1996 issue.

The theory of special relativity developed by Einstein describes what happens if we travel at close to the speed of light. It includes a function called "gamma" that depends on a square root involving v, the speed. If v is complex, the question of Riemann sheets comes up. At least two would be involved and probably more, maybe even an infinite number! When v is complex, the theory also predicts other wonderfully eerie effects. I've used ideas based on that for many stories, including *Primary Inversion*, my first published novel, and "Light and Shadow," the novelette in this anthology. It's been fascinating to play with such fictional extrapolations of the math.

A Harmony of Arches

"The Spacetime Pool" also draws on another of my favorite topics, Fourier analysis. I imagined the Fourier Hall in that story as a great room filled with arches. That idea grew out of my research for another book, *The Veiled Web*, which takes place in Morocco. In doing my research, I fell in love with the gorgeous architecture of Spanish Andalusia and North Africa. I spent hours reading books filled with glossy color pictures of those exquisite buildings. I was also captivated by the repeating nature of the arches, such as in the mosque-cathedral below:

Figure 3: Mezquita de Córdoba, or The Great Mosque of Cordoba, now known as the Catedral de Nuestra Señora de la Asunción

Fourier analysis is used to model periodic functions, so I wondered if it would work for the arches. First I needed an equation that resembled the architecture. I ended up with a pretty good approximation by summing a series of squared sine functions, like this:

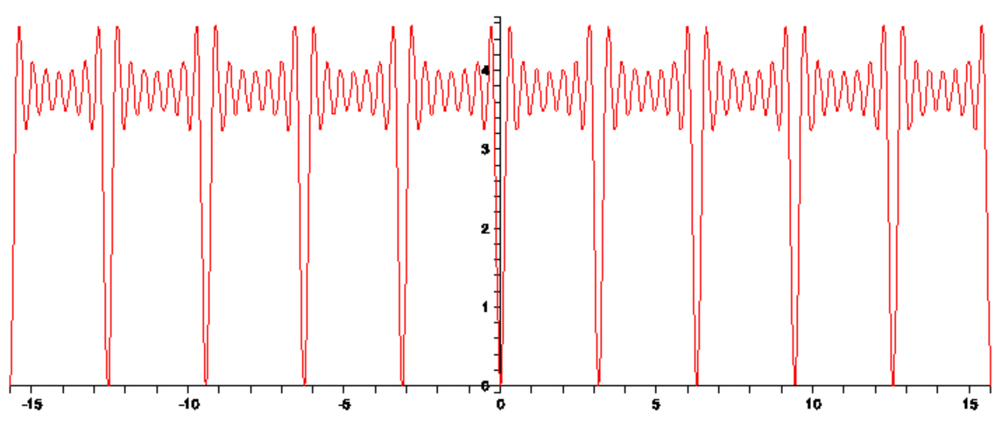

Figure 4: Mathematical model of Moorish arches created by summing sinusoidal functions

Fourier methods can analyze a signal that repeats in time, say an AC current at 60 cycles per second. If we Fourier transform the signal, we get a new function that depends on frequency instead of time. That new function picks out which frequencies contribute to the signal. For the example with AC current, the transform would give a single line at 60 cycles per second. Most functions aren't that simple; more than one frequency contributes to their shape.

Decomposing a function into frequencies is like breaking down a musical chord into its individual notes. Each note has a specific frequency; put them together, they create the harmonic sound we call a chord. Similarly, a Fourier transform decomposes a function into individual terms that each oscillate at a different frequency.

We can also do the transform backward, that is, take a function that depends on frequency into one that depends on time. In "The Spacetime Pool," Janelle transforms the arches of the Fourier Hall into the time domain. I was curious to see what would actually happen with that transform, so I did it on my sinusoidal arches and got the plot below:

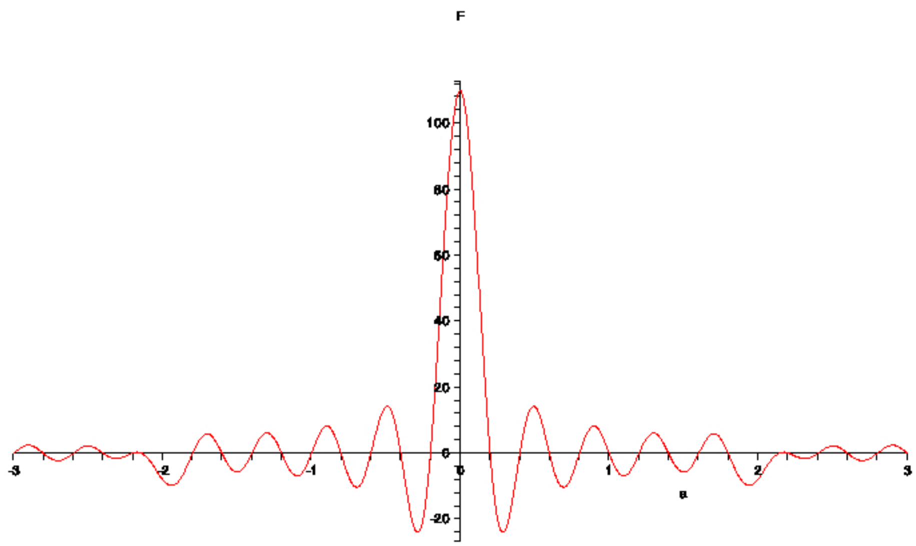

Figure 5: Fourier transform of mathematical model for Moorish arches

That big peak specifies a time that dominates the rest of the function. The question for the story, then, is what does that time mean? "The Spacetime Pool," is actually the first third of a book I'm writing that revolves around the secrets hidden by that enigmatic Fourier Hall. In the novella in this anthology, Janelle has many puzzles to solve, and a few aren't fully resolved. The last two-thirds of the book will reveal the rest of the secrets.

In the novella "Aurora in Four Voices," the Fourier Fount uses individual fountains to mimic the frequency terms in a Fourier series. Each color of the lights bathing the Fount has a specific frequency that corresponds to one of the fountains. When all those graceful arches of water turn on at once, they create a periodic wave bathed in a sparkling rainbow of light. Of course, fountaining water can't behave exactly like a sum of Fourier terms, but I figured it could come close enough to create a beautiful result. I've always wondered what it would look like.

Using math ideas in my fiction is an aspect of writing I find particularly satisfying. It allows me to blend two of my favorite subjects–math and writing–and create something new.